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NOTE ON A PROOF IN CHRYSTAL'S ALGEBRA.

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An analytic proof of the "fundamental theorem of algebra" is given in Chrystal's *Text-book of Algebra*, Part I, 5th Ed., Chap. XII. In view of the character and authority of this admirable treatise it may be worth while to call in question one or two points in this proof.

The argument, in outline, is substantially this: It is first established, in effect, that the rational, integral function $f(w)$ of degree n in w vanishes, if at all, within a circle, S, having a radius R and the origin for center. It is then virtually assumed that within this circle S, $|f(w)|$ has a lower limit L. The further assumption is made that $L>0$, and the argument is arranged to show that this latter assumption involves a contradiction. To do this, a point w is taken within S, such that

$$|f(w)|=L+\varepsilon,\ \varepsilon>0.$$

A neighboring point, $w+h$, is then taken for the purpose of showing that

$$|f(w+h)|<L,$$

when the h is properly chosen. Now, h having been suitably determined, the argument proceeds:

" $\left|\dfrac{f(w+h)}{f(w)}\right|$ will lie between two positive proper fractions, so that

$$\left|\frac{f(w+h)}{f(w)}\right|=1-\mu,$$

where μ is a proper fraction; and we have

$$|f(w+h)|=(1-\mu)\,|f(w)|=(1-\mu)(L+\varepsilon)=L+\varepsilon-\mu(L+\varepsilon) \tag{7}$$

which, since ϵ may be as small as we please, is less than L by a finite amount."

The statement following equation (7) is based apparently on the mutual independence of ε and μ. But ε and μ are not independent, being related through their mutual dependence on w. Both are functions of w, as are also the limits of the interval to which

$$\left|\frac{f(w+h)}{f(w)}\right|$$

i. e., $1-\mu$ is assigned. It is conceivable, therefore, that so long as ε is not zero

$$\varepsilon>\frac{\mu L}{1-\mu},$$

and hence that

$$L+\varepsilon-\mu(L+\varepsilon)>L, \ \varepsilon>0.$$

The argument, therefore, as given in the *Text-book*, would not seem to justify the conclusion that the assumption, $L>0$, involves a contradiction. Hence the proof does not appear to be conclusive.

But, waiving this objection for a moment, it is not shown that the point $w+h$ lies within S. This is necessary. For, $f(w)$ does not vanish on S or without it. Hence, if $|f(w+h)| < L$ when and only when the point $w+h$ is on the circle S or without it, then the conclusion is that $f(w)$ does not vanish within S—the opposite of that desired.

Further, apparently this proof presupposes the continuity of $f(w)$. For, let it be granted that $f(w)$ is not everywhere continuous within or on S. Then there may be a region of discontinuity associated with the points giving rise to the limit value L. If so, it is conceivable, until the contrary is shown, that there exists a constant η, $\eta>0$, such that for any w, $|f(w)| \neq L$, we have $\varepsilon>\eta$. Then "ϵ may not be as small as we please." Hence we cannot affirm that a point $w+h$ exists for which

$$|f(w+h)| < L.$$

Continuity, at least in the neighborhood of the points giving rise to the limit value L, has been assumed. The deduction, therefore, of the continuity of $f(w)$ from this proof, as is made in Cor. 2, seems hardly legitimate.

Two of the objections just raised may be avoided by proceeding somewhat as follows:*

*No attempt is made to give an exact and complete formulation of a proof of "the fundamental theorem of algebra." Nor does the *Text-book* up to the point under consideration furnish a wholly satisfactory basis for the line of reasoning here adopted.

Let

$$f(w) = \sum_{0}^{n} ia_i w^{n-i}, \qquad |a_0| > 0,$$

a_i a complex number, be a polynomial in w which does not vanish on a circle S or without it, S having a radius R and the origin for center. Assume first that $f(w)$ has in S a lower limit L; and second that $L>0$. Then there is in S an unending sequence of values,† w,

(1) $$w_1, w_2, \ldots, w_n, \ldots$$

defining a number or point W within or on S, with which can be associated an unending sequence of positive numbers, ε,

(2) $$\varepsilon_1, \epsilon_2, \ldots, \varepsilon_n, \ldots, \varepsilon_i \overline{>} \varepsilon_{i+1}, \lim_{i \doteq \infty} \varepsilon_i = 0,$$

such that

(3) $$0 < |f(w_i)| - L < \varepsilon_i, \quad i=1, 2, \ldots, \infty.$$

Now employing in part Chrystal's notation,

(4) $$f(w+h) = f(w) + \sum_{1}^{n} {}^j B_j h^j,$$

where B_j is a polynomial in w of degree $n-j$. $B_n = a_0$, is different from zero, and independent of w. It may be shown, after the manner of Chap. V, § 16, that B_j, if it vanish at all, cannot vanish for more than $n-j$ values w. Therefore, there are at most $n(n-1)/2$ values, w, for which some B_j or every B_j vanishes. Some of these values w for which the B's vanish may coincide with W, say τ of them, $0 \overline{<} \tau \overline{<} n(n-1)/2$. Since the number of values, or points, w, for which the B's vanish is finite, it is possible to draw from W as a center a circle S' having on or within it only those vanishing points of the B's coincident with the W, τ in all. The circle S' may lie wholly within or partly without S. Henceforth we are concerned only with the region common to the two circles S and S'. The boundary of this common region we designate by S_1.

Let us further assume that for $w = W$,

†We assume here the continuity of $f(w)$ in the neighborhood of the points giving rise to the limit value L. It would be easy and, perhaps, much simpler to prove, once for all, that every rational integral function of w is continuous everywhere in the finite plane; that its absolute value has a lower limit in every bounded region of this plane; that the absolute value of the function actually assumes this limit value for some point within or on the boundary of this region.

$$B_1 = B_2 = \ldots = B_{m-1} = 0, \quad |B_m| > 0,\ 0 < m \leqq n.$$

Since B_m does not vanish for $w = W$ it does not vanish within or on S_1. Inasmuch as B_j, $j=1, 2, \ldots, n$, is a polynymial in w and $|f(w)| > L$ in S_1, then $B_j/f(w)$ does not become infinite in S_1. Hence there is a positive num- D such that in S_1,

$$\left| \frac{B_j}{f(w)} \right| < D,\ j=1, 2, \ldots, n. \tag{5}$$

In particular, let

$$\left| \frac{B_m}{f(w)} \right| < D_m. \tag{6}$$

Since B_m does not vanish within or on S_1 we assume that within this region $\left| \frac{B_m}{f(w)} \right|$ has a lower limit d_m, $d_m > 0$.

From some value of j on, say $j \overline{\gtrless} k$, all terms w_j of the sequence (1) are in S_1. Only the values w_j, $j \overline{\gtrless} k$, will be considered in what follows.

Now in equation (4) replacing w and h by w_l and h_l, we have

$$f(w_l + h_l) = f(w_l) + \sum_1^n{}^j B_{j,l}\, h_l{}^j, \quad l \overline{\gtrless} k, \tag{7}$$

where the subscript l in $B_{j,l}$ signifies that the B's are now polynomials in w_l. Let us set

$$\frac{B_{j,l}}{f(w_l)} = b_{j,l}\,(\cos a_{j,l} + i\sin a_{j,l}),$$

$$h_l = r(\cos \theta_l + i\sin \theta_l).$$

Then

$$\frac{f(w_l + h_l)}{f(w_l)} = 1 + \sum_1^n{}^j b_{j,l}\, r^j\, \theta_{j,l}, \tag{8}$$

where

$$\Theta_{j,l} = \cos(j\,\theta_l + a_{j,l}) + i\sin(j\,\theta_l + a_{j,l}).$$

We now assign to θ_l values such that

$$\Theta_{m,l} = -1, \quad l \overline{\gtrless} k. \tag{9}$$

Equation (8) becomes, consequently,

$$(10)\qquad \frac{f(w_l+h_l)}{f(w_l)}=1-b_{m,\,l}r^m+\sum_1^n{}^j b_{j,\,l}r^j\,\Theta'_{j,\,l},\quad l\overline{\gtrless}k,\ j\neq m,$$

where the accent on $\Theta'_{j,\,l}$ signifies that θ_l now has the value assigned by equation (9). Hence

$$(11)\qquad 1-b_{m,\,l}-\left|\sum_1^n{}^j b_{j,\,l}r^j\Theta'_{j,\,l}\right|\overline{\lessgtr}\left|\frac{f(w_l+h_l)}{f(w_l)}\right|\overline{\lessgtr}1-b_{m,\,l}r^m$$

$$+\left|\sum_1^n{}^j b_{j,\,l}r^j\,\Theta'_{j,\,l}\right|,\quad l\overline{\gtrless}k,\ j\neq m.$$

By equation (6) and the remark following,

$$(12)\qquad d_m\overline{\lessgtr}b_{m,\,l}<D_m;\ \text{further,}$$

$$(13)\qquad \left|\sum_{m+1}^n{}^j b_{j,\,l}r^j\Theta'_{j,\,l}\right|\overline{\lessgtr}\sum_{m+1}^n{}^j b_{j,\,l}r^j<D\sum_{m+1}^n{}^j r^j.$$

For brevity, let

$$(14)\qquad \phi_l=\left|\sum_1^{m-1}{}^j b_{j,\,l}r^j\Theta'_{j,\,l}\right|.$$

We now take r, $0<r<1$, so that

$$r^{m+1}>r^{m+2}>\ldots>r^n.$$

Hence,

$$(15)\qquad (n-m)r^{m+1}>\sum_{m+1}^n{}^j r^j.$$

The inequalities (12) may now be replaced by

$$(16)\qquad 1-D_mr^m-\phi_l-(n-m)Dr^{m+1}<\left|\frac{f(w_l+h_l)}{f(w_l)}\right|$$

$$<1-d_mr^m+\phi_l+(n-m)Dr^{m+1}.$$

We may now choose r_1, $r_1>0$, so that, $r<r_1$,

$$(17)\qquad 1-D_mr^m-(n-m)Dr^{m+1}>2g,\ g>0,\ \text{and a constant;}$$

and likewise, r_2, $r_2>0$, so that, $r<r_2$,

$$(18)\qquad 1-d_mr^m+(n-m)Dr^{m+1}<1-2f,\ f>0,\ \text{and a constant.}$$

Hence there is a number r_3, $0<r_3<r_1$, $r_3<r_2$, such that, $r \leqq r_3$, the inequalities (17) and (18) are both satisfied.

Let it be assumed that $b_{j,l}$, $j=1, 2, \ldots, (m-1)$, continually approaches zero when w_l approaches W as a limit. Hence we can choose an integer k_1 so that $l>k_1$,

(19) $$|b_{j,l}\,\Theta'_{j,l}| \leqq f'/m,\ j=1, 2, \ldots, m-1,\ 0 \leqq f' < f,\ f' < g,$$

where f' is a constant. Hence

(20) $$\phi_l \leqq f',\ l>k_1.$$

Therefore, by (17), (18), and (20),

(21) $$g < \left| \frac{f(w_l+h_l)}{f(w_l)} \right| < 1-f,\ l>k_1,\ r \leqq r_3.$$

Let

$$\left| \frac{f(w_l+h_l)}{f(w_l)} \right| = 1-\mu_l.$$

Then

(22) $$g<1-\mu_l<1-f,\ l>k_1,\ r \leqq r_3,$$

whence

(23) $$1-g>\mu_l>f.$$

Therefore, μ_l is a positive proper fraction which can approach neither zero nor one as a limit when l approaches infinity. Hence

(24) $$|f(w_l+h_l)| = (1-\mu_l)\,|f(w_l)| = (1-\mu_l)(L+\epsilon)$$
$$= L+\varepsilon-\mu_l(L+\epsilon),\ l > k_1,\ 0 \leqq r \leqq r_3.$$

Now, there is an integer k_2 such that for $l>k_2$ and for r, $0<\delta \leqq r \leqq r_3$, where $\delta<r_3$ and is a constant,

(25) $$L+\varepsilon-\mu(L+\varepsilon)<L.$$

Let us now choose a number r_4, $r_4<r_3$, so that

(26) $$0<r_4<R-|w_l|,\ k_2<l<k_3,\ k_3 \text{ an integer},$$

and further condition δ by the inequality, $\delta<r_4$. Then the inequality (25) holds for every point w_l+h_l such that $k_2<l<k_3$ and $\delta<r<r_4$; further, every such point is in S.

Therefore, L, $L>0$, is not the lower limit of $f(w)$ in S. Hence $L=0$, and $f(w)$ has a root in S.